UNIT–III

11 CHAPTER

INVERSE LAPLACE TRANSFORMS

(SOLUTION OF DIFFERENTIAL EQUATIONS)

11.1 INVERSE LAPLACE TRANSFORMS

If $F(s)$ is the Laplace Transform of a function $f(t)$, then $f(t)$ is known as Inverse Laplace Transform. Now we will discuss how to find $f(t)$ when $F(s)$ is given.

If $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$, where L^{-1} is called the Inverse Laplace Transform operator.

From the application point of view, the Inverse Laplace Transform is very useful.

Inverse Laplace Transform is used in solving differential equations without finding the general solution and arbitrary constants.

11.2 IMPORTANT FORMULAE

1. $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$

3. $L^{-1}\frac{1}{s-a} = e^{at}$

4. $L^{-1}\frac{s}{s^2-a^2} = \cosh at$ *(R.G.P.V., Bhopal, Dec. 2007)*

5. $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$

6. $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

7. $L^{-1}\frac{s}{s^2+a^2} = \cos at$

8. $L^{-1}F(s-a) = e^{at}f(t)$

9. $L^{-1}\frac{1}{(s-a)^2+b^2} = \frac{1}{b}e^{at}\sin bt$

10. $L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$

11. $L^{-1}\frac{1}{(s-a)^2-b^2} = \frac{1}{b}e^{at}\sinh bt$

12. $L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$

13. $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$

14. $L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$

15. $L^{-1}\frac{s^2-a^2}{(s^2+a^2)^2} = t\cos at$

16. $L^{-1}(1) = \delta(t)$

17. $L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{1}{2a}[\sin at + at\cos at]$

18. $L^{-1}\left\{\frac{1}{t}F(t)\right\} = \int_0^t f(t)\,dt$

Example 1. *Show that* $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$. *(U.P., II Semester, Summer 2005)*

Solution. We have to show that $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$.

Now,
$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\Gamma n}$$

So
$$L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{t^{\frac{1}{2}-1}}{\Gamma 1/2} = \frac{t^{-1/2}}{\Gamma 1/2} = \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{1}{\sqrt{\pi t}} \quad \Rightarrow \quad \frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$$ **Proved.**

Example 2. *Find the inverse Laplace Tranform of the following:*

(i) $\frac{1}{s-2}$ (ii) $\frac{1}{s^2-9}$ (iii) $\frac{s}{s^2-16}$

(iv) $\frac{1}{s^2+25}$ (v) $\frac{s}{s^2+9}$ (vi) $\frac{1}{(s-2)^2+1}$

(vii) $\frac{s-1}{(s-1)^2+4}$ (viii) $\frac{1}{(s+3)^2-4}$ (ix) $\frac{s+2}{(s+2)^2-25}$

(x) $\frac{1}{2s-7}$

Solution.

(i) $L^{-1}\frac{1}{s-2} = e^{2t}$

(ii) $L^{-1}\frac{1}{s^2-9} = L^{-1}\frac{1}{3}\cdot\frac{3}{s^2-(3)^2} = \frac{1}{3}\sinh 3t$

(iii) $L^{-1}\frac{s}{s^2-16} = L^{-1}\frac{s}{s^2-(4)^2} = \cosh 4t$

(iv) $L^{-1}\frac{1}{s^2+25} = \frac{1}{5}\frac{5}{s^2+(5)^2} = \frac{1}{5}\sin 5t$

(v) $L^{-1}\frac{s}{s^2+9} = \frac{s}{s^2+(3)^2} = \cos 3t$

(vi) $L^{-1}\frac{1}{(s-2)^2+1} = e^{2t}\sin t$

(vii) $L^{-1}\frac{s-1}{(s-1)^2+4} = e^{t}\cos 2t$

(viii) $L^{-1}\frac{1}{(s+3)^2-4} = \frac{1}{2}\frac{2}{(s+3)^2-(2)^2} = \frac{1}{2}e^{-3t}\sinh 2t$

(ix) $L^{-1}\frac{s+2}{(s+2)^2-25} = L^{-1}\frac{(s+2)}{(s+2)^2-(5)^2} = e^{-2t}\cosh 5t$

(x) $L^{-1}\frac{1}{2s-7} = \frac{1}{2}e^{\frac{7}{2}t}$ $\left[L^{-1}F(as) = \frac{1}{a}f\left(\frac{t}{a}\right)\right]$

Example 3. *Find* $L^{-1}\dfrac{s^2+2s+6}{s^3}$ (M.D.U. 2010)

Solution. Here, we have

$$L^{-1}\frac{s^2+2s+6}{s^3} = L^{-1}\left[\frac{1}{s}+\frac{2}{s^2}+\frac{6}{s^3}\right] = 1+\frac{2t}{1!}+\frac{6}{2!}t^2$$
$$= 1+2t+3t^2$$
Ans.

Example 4. *Find Inverse Laplace Transform of*

(a) $\left\{\dfrac{6}{2s-3}-\dfrac{3+4s}{9s^2-16}+\dfrac{8-6s}{16s^2+9}\right\}$ (b) $\dfrac{2s-5}{9s^2-25}$

(c) $\dfrac{s-2}{6s^2+20}$ (U.P. II. Semester Summer 2001)

Solution.

(a) $$L^{-1}\left\{\frac{6}{2s-3}-\frac{3}{9s^2-16}-\frac{4s}{9s^2-16}+\frac{8}{16s^2+9}-\frac{6s}{16s^2+9}\right\}$$
$$= L^{-1}\left\{\frac{3}{s-\frac{3}{2}}-\frac{\frac{1}{3}}{s^2-\left(\frac{4}{3}\right)^2}-\frac{\frac{4}{9}s}{s^2-\left(\frac{4}{3}\right)^2}+\frac{\frac{1}{2}}{s^2+\left(\frac{3}{4}\right)^2}-\frac{\frac{3}{8}s}{s^2+\left(\frac{3}{4}\right)^2}\right\}$$
$$= L^{-1}\left\{\frac{3}{s-\frac{3}{2}}-\frac{1}{4}\,\frac{\frac{4}{3}}{s^2-\left(\frac{4}{3}\right)^2}-\frac{4}{9}\,\frac{s}{s^2-\left(\frac{4}{3}\right)^2}+\frac{2}{3}\,\frac{\frac{3}{4}}{s^2+\left(\frac{3}{4}\right)^2}-\frac{3}{8}\,\frac{s}{s^2+\left(\frac{3}{4}\right)^2}\right\}$$
$$= 3e^{\frac{3}{2}t}-\frac{1}{4}\sinh\frac{4}{3}t-\frac{4}{9}\cosh\frac{4}{3}t+\frac{2}{3}\sin\frac{3}{4}t-\frac{3}{8}\cos\frac{3}{4}t$$
Ans.

(b) $$L^{-1}\frac{2s-5}{9s^2-25} = L^{-1}\left[\frac{2s}{9s^2-25}-\frac{5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}-\frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$
$$= \frac{2}{9}\cosh\frac{5}{3}t-\frac{1}{3}L^{-1}\left(\frac{\frac{5}{3}}{s^2-\left(\frac{5}{3}\right)^2}\right) = \frac{2}{9}\cosh\frac{5t}{3}-\frac{1}{3}\sin\frac{5t}{3}$$
Ans.

(c) $$L^{-1}\frac{s-2}{6s^2+20} = L^{-1}\frac{s}{6s^2+20}-L^{-1}\frac{2}{6s^2+20} = \frac{1}{6}L^{-1}\frac{s}{s^2+\frac{10}{3}}-\frac{1}{3}L^{-1}\frac{1}{s^2+\frac{10}{3}}$$
$$= \frac{1}{6}L^{-1}\frac{s}{s^2+\frac{10}{3}}-\frac{1}{3}\times\sqrt{\frac{3}{10}}\,L^{-1}\frac{\sqrt{\frac{10}{3}}}{s^2+\frac{10}{3}} = \frac{1}{6}\cos\sqrt{\frac{10}{3}}\,t-\frac{1}{\sqrt{30}}\sin\sqrt{\frac{10}{3}}\,t$$
Ans.

Example 5. *Find the inverse Laplace transform of following functions:* $\dfrac{14s+10}{49s^2+28s+13}$. [U.P., II Semester, 2007]

Solution. The given function can be written as

$$\frac{14s+10}{49s^2+28s+13} = \frac{14s+10}{(7s+2)^2+9} = \frac{14\left(s+\frac{2}{7}\right)+6}{49\left(s+\frac{2}{7}\right)^2+9}$$

$$\therefore\quad L^{-1}\left(\frac{14s+10}{49s^2+28s+13}\right) = L^{-1}\left[\frac{14\left(s+\frac{2}{7}\right)+6}{49\left(s+\frac{2}{7}\right)^2+9}\right] = e^{-\frac{2t}{7}}L^{-1}\left(\frac{14s+6}{49s^2+9}\right)$$
$$= e^{-\frac{2}{7}t}L^{-1}\frac{14}{49}\left(\frac{s+\frac{6}{14}}{s^2+\frac{9}{49}}\right)$$
$$= e^{-\frac{2t}{7}}\left[\frac{14}{49}L^{-1}\left(\frac{s}{s^2+\frac{9}{49}}\right)+\left(\frac{14}{49}\right)\left(\frac{6}{14}\right)L^{-1}\left(\frac{1}{s^2+\frac{9}{49}}\right)\right]$$
$$= e^{-\frac{2t}{7}}\left[\frac{2}{7}\cos\frac{3}{7}t+\frac{6}{49}\cdot\frac{7}{3}\sin\frac{3}{7}t\right]$$
$$= \frac{2}{7}e^{-\frac{2t}{7}}\left(\cos\frac{3}{7}t+\sin\frac{3}{7}t\right)$$
Ans.

EXERCISE 11.1

Find the **Inverse Laplace Transform** of the following:

1. $\dfrac{3s-8}{4s^2+25}$ **Ans.** $\dfrac{3}{4}\cos\dfrac{5t}{2}-\dfrac{4}{5}\sin\dfrac{5t}{2}$
2. $\dfrac{3\left(s^2-2\right)^2}{2s^5}$ **Ans.** $\dfrac{3}{2}-3t^2+\dfrac{1}{2}t^4$
3. $\dfrac{2s-5}{4s^2+25}+\dfrac{4s-18}{9-s^2}$ **Ans.** $\dfrac{1}{2}\left(\cos\dfrac{5t}{2}-\sin\dfrac{5t}{2}\right)-4\cosh 3t+6\sinh 3t$
4. $\dfrac{5s-10}{9s^2-16}$ **Ans.** $\dfrac{5}{9}\cosh\dfrac{4}{3}t-\dfrac{5}{6}\sinh\dfrac{4}{3}t$
5. $\dfrac{1}{4s}+\dfrac{16}{1-s^2}$ **Ans.** $\dfrac{1}{4}-16\sinh t$
6. $L^{-1}\left\{\dfrac{1}{s^n}\right\}$ exist only when the value of n is :

   (i) Negative integer (ii) Positive integer
   (iii) Zero (iv) None of these **Ans.** (ii) (U.P. II Semester, 2010)

11.3 MULTIPLICATION BY s

$$L^{-1}[s\,F(s)] = \frac{d}{dt} f(t) + f(0)\,\delta(t)$$

Example 6. *Find the Inverse Laplace Transform of* (*i*) $\frac{s}{s^2+1}$ (*ii*) $\frac{s}{4s^2-25}$ (*iii*) $\frac{3s}{2s+9}$

Solution.

(*i*) $L^{-1}\frac{1}{s^2+1} = \sin t$

$L^{-1}\frac{s}{s^2+1} = \frac{d}{dt}(\sin t) + \sin(0)\,\delta(t) = \cos t$ **Ans.**

(*ii*) $L^{-1}\frac{1}{4s^2-25} = \frac{1}{4}L^{-1}\frac{1}{s^2-\frac{25}{4}} = \frac{1}{4}\cdot\frac{2}{5}L^{-1}\frac{\frac{5}{2}}{s^2-\left(\frac{5}{2}\right)^2} = \frac{1}{10}\sinh\frac{5}{2}t$

$L^{-1}\frac{s}{4s^2-25} = \frac{1}{10}\frac{d}{dt}\sinh\frac{5}{2}t + \frac{1}{10}\sinh\frac{5}{2}(0)\,\delta(t)$

$= \frac{1}{10}\left(\frac{5}{2}\right)\cosh\frac{5}{2}t = \frac{1}{4}\cosh\frac{5}{2}t$ **Ans.**

(*iii*) $L^{-1}\frac{3}{2s+9} = \frac{3}{2}L^{-1}\frac{1}{s+\frac{9}{2}} = \frac{3}{2}e^{-\frac{9}{2}t}$

$L^{-1}\frac{3s}{2s+9} = \frac{3}{2}\frac{d}{dt}\left(e^{-\frac{9}{2}t}\right) + \frac{3}{2}e^{-\frac{9}{2}(0)\delta(t)} = \frac{3}{2}\left(-\frac{9}{2}\right)e^{-\frac{9}{2}t} + \frac{3}{2}$

$= -\frac{27}{4}e^{-\frac{9}{2}t} + \frac{3}{2}$ **Ans.**

EXERCISE 11.2

Find the Inverse Laplace Transform of the following:

1. $\frac{s}{s+5}$ **Ans.** $-5\,e^{-5t}+1$
2. $\frac{2s}{3s+6}$ **Ans.** $\frac{2}{3}\left[-2e^{-2t}+1\right]$
3. $\frac{s}{2s^2-1}$ **Ans.** $\frac{1}{2}\cosh\frac{t}{2}$
4. $\frac{s^2}{s^2+a^2}$ **Ans.** $-a\sin at+1$
5. $\frac{s^2+4}{s^2+9}$ **Ans.** $-\frac{5}{3}\sin 3t+1$

11.4 DIVISION BY s (MULTIPLICATION BY $\frac{1}{s}$)

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t\left[L^{-1}[F(s)]\right]dt = \int_0^t f(t)\,dt$$

Example 7. *Find the Inverse Laplace Transform of*

(*i*) $\frac{1}{s(s+a)}$ (*ii*) $\frac{1}{s(s^2+1)}$ (*iii*) $\frac{s^2+3}{s(s^2+9)}$

Solution.

(*i*) $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt = \int_0^t e^{-at}\,dt = \left[\frac{e^{-at}}{-a}\right]_0^t$

$= \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1}{a}\left[1-e^{-at}\right]$ **Ans.**

(*ii*) $L^{-1}\frac{1}{s^2+1} = \sin t$

$L^{-1}\frac{1}{s}\left(\frac{1}{s^2+1}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+1}\right)dt = \int_0^t \sin t\,dt = [-\cos t]_0^t = -\cos t + 1$ **Ans.**

(*iii*) $L^{-1}\frac{s^2+3}{s(s^2+9)} = L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right] = L^{-1}\left[\frac{1}{s} - \frac{6}{s(s^2+9)}\right]$

$= 1 - 2\int_0^t \sin 3t\,dt = 1 + 2\times\frac{1}{3}[\cos 3t]_0^t = 1 + \frac{2}{3}\cos 3t - \frac{2}{3}$

$= \frac{2}{3}\cos 3t + \frac{1}{3} = \frac{1}{3}[2\cos 3t + 1]$ **Ans.**

EXERCISE 11.3

Find the Inverse Laplace Transform of the following:

1. $\frac{1}{2s(s-3)}$ **Ans.** $\frac{1}{2}\left[\frac{e^{3t}}{3}-1\right]$
2. $\frac{1}{s(s+2)}$ **Ans.** $\frac{1-e^{-2t}}{2}$
3. $\frac{1}{s(s^2-16)}$ **Ans.** $\frac{1}{16}[\cosh 4t-1]$
4. $\frac{1}{s(s^2+a^2)}$ **Ans.** $\frac{1-\cos at}{a^2}$
5. $\frac{s^2+2}{s(s^2+4)}$ **Ans.** $\cos^2 t$
6. $\frac{1}{s^2(s+1)}$ **Ans.** $t-1+e^{-t}$
7. $\frac{1}{s^3(s^2+1)}$ **Ans.** $\frac{t^2}{2}+\cos t-1$

11.5 FIRST SHIFTING PROPERTY

If $L^{-1}F(s) = f(t)$, then $L^{-1}F(s+a) = e^{-at}L^{-1}[F(s)]$

Example 8. *Find the Inverse Laplace Transform of*

(*i*) $\frac{1}{(s+2)^5}$ (*ii*) $\frac{s}{s^2+4s+13}$ (*iii*) $\frac{1}{9s^2+6s+1}$

Solution.

(*i*) $L^{-1}\dfrac{1}{s^5}=\dfrac{t^4}{4!}$

then $L^{-1}\dfrac{1}{(s+2)^5}=e^{-2t}\cdot\dfrac{t^4}{4!}$ **Ans.**

(*ii*) $$L^{-1}\left(\frac{s}{s^2+4s+13}\right)=L^{-1}\frac{s+2-2}{(s+2)^2+(3)^2}=L^{-1}\frac{s+2}{(s+2)^2+(3)^2}-L^{-1}\frac{2}{(s+2)^2+(3)^2}$$

$$=e^{-2t}L^{-1}\frac{s}{s^2+3^2}-e^{-2t}L^{-1}\frac{2}{3}\left(\frac{3}{s^2+3^2}\right)$$

$$=e^{-2t}\cos 3t-\frac{2}{3}e^{-2t}\sin 3t$$ **Ans.**

(*iii*) $$L^{-1}\frac{1}{9s^2+6s+1}=L^{-1}\frac{1}{(3s+1)^2}=\frac{1}{9}L^{-1}\frac{1}{\left(s+\frac{1}{3}\right)^2}=\frac{1}{9}e^{-t/3}L^{-1}\frac{1}{s^2}$$

$$=\frac{1}{9}e^{-t/3}\,t=\frac{t\,e^{-t/3}}{9}$$ **Ans.**

Example 9. *Find the Inverse Laplace Transform of*

$$\frac{5s+3}{(s-1)(s^2+2s+5)}$$ (*U.P. II Semester Summer* 2005)

Solution. $$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$$

Let $$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

$$5s+3=A(s^2+2s+5)+(Bs+C)(s-1)$$

$$5s+3=s^2(A+B)+s(2A-B+C)+(5A-C)$$

Comparing the coefficients of s^2, s and constant, we get

$$A+B=0 \quad \text{... (1)}$$

$$2A-B+C=5 \quad \text{... (2)}$$

$$5A-C=3 \quad \text{... (3)}$$

On adding equations (1) and (2), we have

$$3A+C=5 \quad \text{... (4)}$$

Adding equations (3) and (4), we get

$$8A=8 \quad\Rightarrow\quad A=1$$

Putting A = 1 in (3), we get

$$C=2$$

Putting $A=1$, $C=2$ in (2), we get

$$B=-1$$

Thus $$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{1}{s-1}+\frac{-s+2}{s^2+2s+5}=\frac{1}{s-1}-\frac{s-2}{(s+1)^2+2^2}$$

$$=\frac{1}{s-1}-\frac{s+1}{(s+1)^2+2^2}+\frac{3}{(s+1)^2+2^2}$$

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=L^{-1}\left\{\frac{1}{s-1}\right\}+L^{-1}\left\{\frac{3}{(s+1)^2+2^2}\right\}-L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$$

$$=e^t+3e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\}-e^{-t}L^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$=e^t+3e^{-t}\cdot\frac{1}{2}\sin 2t-e^{-t}\cos 2t$$ **Ans.**

EXERCISE 11.4

Obtain the Inverse Laplace Transform of the following:

1. $\dfrac{s+8}{s^2+4s+5}$ **Ans.** $e^{-2t}(\cos t+6\sin t)$
2. $\dfrac{s}{(s+3)^2+4}$ **Ans.** $e^{-3t}(\cos 2t-1.5\sin 2t)$
3. $\dfrac{s}{(s+7)^4}$ **Ans.** $e^{-7t}\dfrac{t^2}{6}(3-7t)$
4. $\dfrac{s+2}{s^2-2s-8}$ **Ans.** $e^{-t}(\cosh 3t+\sinh 3t)$
5. $\dfrac{s}{s^2+6s+25}$ (*GBTU, 2011*) **Ans.** $e^{-3t}\left[\cos 4t-\dfrac{3}{4}\sin 4t\right]$
6. $\dfrac{1}{2(s-1)^2+32}$ **Ans.** $\dfrac{e^t}{8}\sin 4t$
7. $\dfrac{s-4}{4(s-3)^2+16}$ **Ans.** $\dfrac{1}{4}e^{3t}\cos 2t-\dfrac{1}{8}e^{3t}\sin 2t$

11.8 SECOND SHIFTING PROPERTY

$$L^{-1}\left[e^{-as}F(s)\right]=f(t-a)\,u(t-a)$$

Example 10. *Obtain Inverse Laplace Transform of*

(*i*) $\dfrac{e^{-\pi s}}{(s+3)}$ (*ii*) $\dfrac{e^{-s}}{(s+1)^3}$

Solution.

(*i*) $$L^{-1}\frac{1}{s+3}=e^{-3t}$$

$$L^{-1}\frac{e^{-\pi s}}{s+3}=e^{-3(t-\pi)}\,u(t-\pi)$$ **Ans.**

(*ii*) $$L^{-1}\frac{1}{s^3}=\frac{t^2}{2!}\quad\Rightarrow\quad L^{-1}\frac{1}{(s+1)^3}=e^{-t}\frac{t^2}{2!}$$

$$L^{-1}\frac{e^{-s}}{(s+1)^3}=e^{-(t-1)}\cdot\frac{(t-1)^2}{2!}\cdot u(t-1)$$ **Ans.**

Example 11. *Evaluate*

$$L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]$$

(*U.P. II Semester, Summer 2002*)

Solution. $L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]=L^{-1}\left[\frac{e^{-s}}{s^2}-\frac{3e^{-3s}}{s^2}\right]$ ...(1)

We know that $L\,[u\,(t-a)]=\frac{e^{-as}}{s}$

and $L\,[(t-a)\,u(t-a)]=\frac{e^{-as}}{s^2}$

Using these results in (1), we get

$$\therefore \quad L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]=(t-1)\,u\,(t-1)-3(t-3)\,u\,(t-3)$$ **Ans.**

Example 12. *Find the Inverse Laplace Transform of*

$$\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$$

in terms of unit step functions.

Solution. $L^{-1}\frac{\pi}{s^2+\pi^2}=\sin \pi t$

$$L^{-1}\left[e^{-s}\frac{\pi}{s^2+\pi^2}\right]=\sin\pi(t-1).\,u(t-1)=-\sin(\pi t).\,u(t-1) \quad ...(1)$$

and $L^{-1}\frac{s}{s^2+\pi^2}=\cos\pi t$

$$L^{-1}\left[e^{-s/2}\frac{s}{s^2+\pi^2}\right]=\cos\pi\left(t-\frac{1}{2}\right).\,u\left(t-\frac{1}{2}\right)=\sin\pi t\,.\,u\left(t-\frac{1}{2}\right) \quad ...(2)$$

On adding (1) and (2), we get

$$L^{-1}\left[\frac{e^{-s/2}\,s+e^{-s}.\,\pi}{s^2+\pi^2}\right]=\sin(\pi t).\,u\left(t-\frac{1}{2}\right)-\sin(\pi t).\,u(t-1)$$

$$=\sin\pi t\left[u\left(t-\frac{1}{2}\right)-u\,(t-1)\right]$$ **Ans.**

Example 13. *Find the value of*

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

Solution. We know that

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$$

Differentiating w.r.t. a, we get

$$L^{-1}\left\{\frac{d}{da}\left(\frac{1}{s^2+a^2}\right)\right\}=\frac{d}{da}\left(\frac{\sin at}{a}\right)$$

$$\Rightarrow \quad L^{-1}\left\{\frac{-2a}{(s^2+a^2)^2}\right\}=-\frac{1}{a^2}\sin at+\frac{t}{a}\cos at$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}\sin at-\frac{t}{2a^2}\cos at$$

$$=\frac{1}{2a^3}(\sin at-at\cos at)$$ **Ans.**

EXERCISE 11.5

Obtain Inverse Laplace Transform of the following:

1. $\frac{e^{-s}}{(s+2)^3}$ **Ans.** $e^{-(t-2)}\frac{(t-2)^2}{2}u(t-2)$
2. $\frac{e^{-2s}}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-(t-2)}\,|\,1-\cos(t-2)\,|\,u(t-2)$
3. $\frac{e^{-s}}{\sqrt{s+1}}$ **Ans.** $\frac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}.\,u(t-1)$
4. $\frac{e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}}{s^2+1}$ **Ans.** $\cot t\left[u\left(t-\frac{3\pi}{2}\right)-u\left(t-\frac{\pi}{2}\right)\right]$
5. $\frac{e^{-4s}(s+2)}{s^2+4s+5}$ **Ans.** $e^{-2(t-u)}\cos(t-u)\,u(t-4)$
6. $\frac{e^{-as}}{s^2}$ **Ans.** $f(t)=t-a$, when $t>a$; $=0$, when $t<a$
7. $\frac{e^{-\pi s}}{s^2+1}$ **Ans.** $-\sin t.\,u(t-\pi)$

11.7 INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

$$L^{-1}\left[\frac{d}{ds}F(s)\right]=-t\,L^{-1}[F(s)]=-t\,f(t) \quad\Rightarrow\quad \boxed{L^{-1}[F(s)]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Example 14. *Find* $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$. (*Uttarakhand, II Semester, June 2010, 2009, 2007*)

Solution. $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right]$

$$=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+1)-\frac{d}{ds}\log(s-1)\right]$$

$$=-\frac{1}{t}L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right]$$

$$=-\frac{1}{t}\left[e^{-t}-e^{t}\right]=\frac{1}{t}\left[e^{t}-e^{-t}\right]$$ **Ans.**

Example 15. *Find the Inverse Laplace Transform of* $F(s) = \log \dfrac{s+a}{s+b}$

(U.P., II Semester, Summer 2003)

Solution. $L^{-1} \log\left(\dfrac{s+a}{s+b}\right) = -\dfrac{1}{t} L^{-1}\left[\dfrac{d}{ds}\log\dfrac{s+a}{s+b}\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log(s+a) - \frac{d}{ds}\log(s+b)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= -\frac{1}{t}\left[e^{-at} - e^{-bt}\right] = \frac{1}{t}\left(e^{-bt} - e^{-at}\right)$$

Ans.

Example 16. *Obtain the Inverse Laplace Transform of* $\log \dfrac{s^2-1}{s^2}$

Solution. $L^{-1}\left[\log\dfrac{s^2-1}{s^2}\right] = -\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\log\dfrac{s^2-1}{s^2}\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log\left(s^2-1\right) - 2\log s\right\}\right] = -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1} - \frac{2}{s}\right] = -\frac{1}{t}[2\cosh t - 2]$$

$$= \frac{2}{t}[1 - \cosh t]$$

Ans.

Example 17. *Find the function whose Laplace transform is*

$\log\left(1+\dfrac{1}{s}\right)$. *(U.P., II Semester, June 2007)*

Solution. $L^{-1}\left[\log\left(1+\dfrac{1}{s}\right)\right] = \dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\log\left(\dfrac{s+1}{s}\right)\right]$

$$= -\frac{1}{t}L^{-1}\left[\left(\frac{s}{s+1}\right)\left(-\frac{1}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[-\frac{1}{s(s+1)}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s}\right] \quad \text{(Partial fraction)}$$

$$= -\frac{1}{t}[e^{-t} - 1] = \frac{1}{t}[1 - e^{-t}]$$

Ans.

Example 18. *Find the inverse Laplace transform of*

$\tan^{-1}\left(\dfrac{2}{s^2}\right)$

(Q.Bank U.P.T.U. 2001)

Solution. Here, we have

$$L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\frac{2}{s^2}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+\frac{4}{s^4}}\left(\frac{4}{s^3}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{s^4}{s^4+4}\left(-\frac{4}{s^3}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right]$$

$$= \frac{4}{t}L^{-1}\left[-\frac{1}{4}\frac{1}{(s^2+2s+2)} + \frac{1}{4}\frac{1}{(s^2-2s+2)}\right] \quad \left(\text{By partial fraction}\right)$$

$$= \frac{1}{t}L^{-1}\left[-\frac{1}{(s^2+2s+2)} + \frac{1}{(s^2-2s+2)}\right]$$

$$= \frac{1}{t}L^{-1}\left[-\frac{1}{(s+1)^2+1} + \frac{1}{(s-1)^2+1}\right]$$

$$= \frac{1}{t}\left[-e^{-t}\sin t + e^{t}\sin t\right]$$

$$= \frac{\sin t}{t}\left[e^{t} - e^{-t}\right]$$

Ans.

Example 19. *Find Inverse Laplace Transform of* $\tan^{-1}\dfrac{1}{s}$.

Solution. $L^{-1}\left(\tan^{-1}\dfrac{1}{s}\right) = -\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\tan^{-1}\dfrac{1}{s}\right]$ *(M.D.U., 2010)*

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(-\frac{1}{s^2}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{1}{1+s^2}\right] = \frac{\sin t}{t}$$

Ans.

Example 20. *Find* $L^{-1}\left[\tan^{-1}(1+s)\right]$ *(M.D.U. 2010)*

Solution. $L^{-1}\left[\tan^{-1}(1+s)\right] = -\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\tan^{-1}(1+s)\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+(s+1)^2}\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -\frac{1}{t}e^{-t}\sin t$$

Ans.

Example 21. *Find the inverse Laplace transform of*

$$\cot^{-1}\left(\frac{s}{2}\right)$$

(Q.Bank U.P. 2001)

Solution.

Let $$L^{-1}\left[\cot^{-1}\left(\frac{s}{2}\right)\right] = f(t)$$

$$\Rightarrow \quad L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right] = -t\,f(t)$$

$$\Rightarrow \quad L^{-1}\left[\frac{-1}{1+\frac{s^2}{4}}\cdot\frac{1}{2}\right] = -t\,f(t)$$

$$\Rightarrow \quad L^{-1}\left[\frac{2}{s^2+2}\right] = t\,f(t)$$

$$\Rightarrow \quad \sin 2t = t\,f(t)$$

$$\Rightarrow \quad f(t) = \frac{1}{t}\sin 2t$$

Ans.

Example 22. *Obtain the Inverse Laplace Transform of* $\cot^{-1}\left(\frac{s+3}{2}\right)$

(U. P., II Semester, Summer 2002)

Solution. We know that $L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

$$\therefore \quad L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{-\frac{1}{2}}{1+\left(\frac{s+3}{2}\right)^2}\right\}$$

$$= \frac{1}{2t}L^{-1}\left\{\frac{4}{4+(s+3)^2}\right\}$$

$$= \frac{1}{t}L^{-1}\left\{-\frac{2}{2^2+(s+3)^2}\right\}$$

$$= \frac{1}{t}e^{-3t}L^{-1}\left(\frac{2}{2^2+s^2}\right)$$

$$= \frac{e^{-3t}}{t}\sin 2t$$

Ans.

Example 23. *Find the inverse Laplace transform of* $\frac{2as}{(s^2+a^2)^2}$

Solution. $$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$$

$$L^{-1}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right] = -t\sin at$$

$$\Rightarrow \quad L^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\} = -t\sin at$$

$$\Rightarrow \quad L^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\} = t\sin at$$

Ans.

Example 24. *Find the inverse Laplace transform of* $\frac{s^2-a^2}{(s^2+a^2)^2}$

Solution. We know that

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\therefore \quad L^{-1}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right] = -t\cos at$$

$$\Rightarrow \quad L^{-1}\left\{\frac{(s^2+a^2).1-s(2s)}{(s^2+a^2)}\right\} = -t\cos at$$

$$\Rightarrow \quad L^{-1}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = -t\cos at$$

$$\therefore \quad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$$

Ans.

EXERCISE 11.6

Obtain Inverse Laplace Transform of the following:

1. $\log\left(1+\frac{\omega^2}{s^2}\right)$ **Ans.** $-\frac{2}{t}\cos\omega t+2$ 2. $\log\left(1+\frac{1}{s^2}\right)$ **Ans.** $\frac{2}{t}[1-\cos\omega t]$
3. $\frac{s}{1+s^2+s^4}$ **Ans.** $\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\sinh\frac{t}{2}$
4. $\frac{s+1}{(s^2+6s+13)^2}$ **Ans.** $\frac{e^{-3t}}{8}[2t\sin 2t+2t\cos 2t-\sin 2t]$
5. $\frac{s}{(s^2+a^2)^2}$ **Ans.** $\frac{t\sin at}{2a}$ 6. $s\log\frac{s}{\sqrt{s^2+1}}+\cot^{-1}s$ **Ans.** $\frac{1-\cos t}{t^2}$
7. $\frac{1}{2}\log\left\{\frac{s^2+b^2}{(s-a)^2}\right\}$ **Ans.** $\frac{e^{-at}-\cos bt}{t}$

11.8 INVERSE LAPLACE TRANSFORM OF INTEGRALS

$$L^{-1}\left[\int_s^\infty F(s)\,ds\right]=\frac{f(t)}{t}=\frac{1}{t}L^{-1}\left[F(s)\right] \quad \text{or} \quad \boxed{L^{-1}\left[F(s)\right]=t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]}$$

Example 25. *Obtain* $L^{-1}\dfrac{2s}{(s^2+1)^2}$

Solution. $$L^{-1}\frac{2s}{(s^2+1)^2}=t\,L^{-1}\int_s^\infty\frac{2s\,ds}{(s^2+1)^2}=t\,L^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty=t\,L^{-1}\left[-0+\frac{1}{s^2+1}\right]$$
$$=t\sin t$$ **Ans.**

11.9 PARTIAL FRACTIONS METHOD

Example 26. *Find the Inverse Laplace Transforms of* $\dfrac{1}{s^2-5s+6}$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right]=L^{-1}\left[\frac{1}{s-3}-\frac{1}{s-2}\right]$$
$$=L^{-1}\left(\frac{1}{s-3}\right)-L^{-1}\left(\frac{1}{s-2}\right)=e^{3t}-e^{2t}$$ **Ans.**

Example 27. *Find the Inverse Laplace Transforms of* $\dfrac{s+1}{s^2-6s+25}$

(U.P., II Semester 2010)

Solution. $$L^{-1}\left(\frac{s+1}{s^2-6s+25}\right)=L^{-1}\left[\frac{s+1}{(s-3)^2+(4)^2}\right]=L^{-1}\left[\frac{s-3+4}{(s-3)^2+(4)^2}\right]$$
$$=L^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right]+L^{-1}\left[\frac{4}{(s-3)^2+(4)^2}\right]$$
$$=e^{3t}\cos 4t+e^{3t}\sin 4t.$$ **Ans.**

Example 28. *Find the inverse Laplace transform of*

$$\frac{s^3}{s^4-a^4}$$ *(Q. Bank U.P. 2007)*

Solution. Here, we have

$$L^{-1}\left(\frac{s^3}{s^4-a^4}\right)=L^{-1}\left[s\left\{\frac{s^2}{(s^2-a^2)(s^2+a^2)}\right\}\right]$$
$$=L^{-1}\left[\frac{s}{2}\left(\frac{1}{s^2-a^2}+\frac{1}{s^2+a^2}\right)\right]$$
$$=\frac{1}{2}L^{-1}\left(\frac{s}{s^2-a^2}+\frac{s}{s^2+a^2}\right) \quad \text{(By partial fraction)}$$
$$=\frac{1}{2}(\cosh at+\cos at)$$ **Ans.**

Example 29. *Find the Inverse Laplace Transforms of* $\dfrac{s+4}{s(s-1)(s^2+4)}$.

Solution. Let us first resolve $\dfrac{s+4}{s(s-1)(s^2+4)}$ into partial fractions.

$$\frac{s+4}{s(s-1)(s^2+4)}\equiv\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+4} \qquad \ldots(1)$$
$$s+4\equiv A(s-1)(s^2+4)+Bs(s^2+4)+(Cs+D)s(s-1)$$

Putting $s=0$, we get $4=-4A \Rightarrow A=-1$

Putting $s=1$, we get $5=B.1.(1+4) \Rightarrow B=1$

Equating the coefficients of s^3 on both sides of (1), we have

$$0=A+B+C \Rightarrow 0=-1+1+C \Rightarrow C=0.$$

Equating the coefficients of s on both sides of (1), we get

$$1=4A+4B-D \Rightarrow 1=-4+4-D \Rightarrow D=-1.$$

On putting the values of A, B, C, D in (1), we get

$$\frac{s+4}{s(s-1)(s^2+4)}=-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}$$

$$\therefore L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right]=L^{-1}\left[-\frac{1}{s}+\frac{1}{s-1}-\frac{1}{s^2+4}\right]$$
$$=-L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$
$$=-1+e^t-\frac{1}{2}\sin 2t.$$ **Ans.**

Example 30. *Find the inverse Laplace transform of*

$$\frac{1}{s^4+4}$$ [U.P., II Semester, (SUM) 2007]

Solution. Here, we have

$$s^4+4=(s^2+2)^2-(2s)^2=(s^2-2s+2)(s^2+2s+2)$$
$$\frac{1}{s^4+4}=\frac{1}{(s^2-2s+2)(s^2+2s+2)}$$
$$=\frac{1}{4s}\left[\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right] \quad \left(\text{By partial fraction}\right) \ldots(1)$$

Now, $$L^{-1}\left(\frac{1}{s^2-2s+2}\right)=L^{-1}\left[\frac{1}{(s-1)^2+1}\right]=e^t\sin t$$

and $$L^{-1}\left(\frac{1}{s^2+2s+2}\right)=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}\sin t$$

$$\therefore \quad \frac{1}{4}L^{-1}\left(\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right) = \frac{1}{4}(e^t-e^{-t})\sin t$$

Hence,

$$L^{-1}\left[\frac{1}{4s}\left(\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right)\right] = \frac{1}{4}\int_0^t (e^t-e^{-t})\sin t\,dt$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{4}\left[\frac{e^t}{2}(\sin t-\cos t)-\frac{e^{-t}}{2}(-\sin t-\cos t)\right]$$

$$= \frac{1}{4}\left[\sin t\left(\frac{e^t+e^{-t}}{2}\right)-\cos t\left(\frac{e^t-e^{-t}}{2}\right)\right]$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{4}[\sin t\cosh t-\cos t\sinh t]$$ **Ans.**

Example 31. *Find the inverse Laplace transform of* $\frac{s}{s^4+4a^4}$

Solution.

$$s^4+4a^4 = (s^2+2a^2)^2-(2as)^2 = (s^2-2as+2a^2)(s^2+2as+2a^2)$$
$$= \{(s-a)^2+a^2\}\{(s+a)^2+a^2\}$$

$$\frac{s}{s^4+4a^4} = \frac{s}{\{(s-a)^2+a^2\}\{(s+a)^2+a^2\}}$$

$$= \frac{1}{4a}\left[\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right] \quad \text{(By partial fraction)}$$

$$\therefore \quad L^{-1}\left(\frac{s}{s^4+4a^4}\right) = \frac{1}{4a}\left[L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}-L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{a}e^{at}\sin at-e^{-at}\frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a^2}\sin at\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2a^2}\sin at\sinh at.$$ **Ans.**

Example 32. *Find the Inverse Laplace Transform of* $\frac{e^{-2\pi s}}{s(s^2+1)}$ *(GBTU 2012)*

Solution. Here, we have $L^{-1}\left[\frac{e^{-2\pi s}}{s(s^2+1)}\right] = L^{-1}\left[\frac{e^{-2\pi s}}{s}-\frac{e^{-2\pi s}}{s^2+1}\right]$

$$= L^{-1}\frac{e^{-2\pi s}}{s}-L^{-1}\frac{e^{-2\pi s}}{s^2+1} = 1.4\,(t-2\pi)-\sin(t-2\pi)\,u\,(t-2\pi)$$ **Ans.**

Example 32a. *Find the Inverse Laplace Transform of* $\frac{e^{-cs}}{s^2(s+a)}$, $c > 0$.

(U.P. II Semester, Summer 2002)

Solution. We have,

$$L^{-1}\left[\frac{e^{-cs}}{s^2(s+a)}\right] = L^{-1}\left[-\frac{e^{-cs}}{a^2 s}+\frac{e^{-cs}}{as^2}+\frac{e^{-cs}}{a^2(s+a)}\right] \quad \text{(By Partial fractions)}$$

$$= L^{-1}\left[\left(\frac{-1}{a^2}\frac{e^{-cs}}{s}\right)+\left(\frac{1}{a}\right)\frac{e^{-cs}}{s^2}+\left(\frac{1}{a^2}\right)\frac{e^{-c(s+a)}}{e^{-ca}(s+a)}\right]$$

$$= -\frac{1}{a^2}u(t-c)+\frac{1}{a}(t-c)\,u(t-c)+\frac{1}{a^2e^{-ca}}e^{at}\,u(t-c)$$

$$= u(t-c)\left[\frac{-1}{a^2}+\frac{1}{a}(t-c)+\frac{1}{a^2}e^{a(c+t)}\right]$$, where $u(t-c)$ is unit step function.

Ans.

Example 33. *Find the Inverse Laplace Transform of* $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{a^2}{a^2-b^2}\cdot\frac{1}{s^2+a^2}-\frac{b^2}{a^2-b^2}\cdot\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}L^{-1}\left[\frac{a^2}{s^2+a^2}-\frac{b^2}{s^2+b^2}\right] = \frac{1}{a^2-b^2}\left[a^2\left(\frac{1}{a}\sin at\right)-b^2\left(\frac{1}{b}\sin bt\right)\right]$$

$$= \frac{1}{a^2-b^2}[\,a\sin at-b\sin bt\,]$$ **Ans.**

Note: This question is also solved by using the Convolution Theorem as an example 37.

EXERCISE 11.7

Find the Inverse Laplace Transforms of the follwoing by partial fractions method:

1. $\frac{1}{s^2-7s+12}$ **Ans.** $e^{4t}-e^{3t}$
2. $\frac{s+2}{s^2-4s+13}$ **Ans.** $e^{2t}\cos 3t+\frac{4}{3}e^{2t}\sin 3t$
3. $\frac{3s+1}{(s-1)(s^2+1)}$ **Ans.** $e^t-2\cos t+\sin t$
4. $\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ **Ans.** $2e^{-t}+5e^{2t}-\frac{3}{2}e^{t/2}$
5. $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$ **Ans.** $\frac{1}{2}e^t-e^{2t}+\frac{5}{2}e^{3t}$
6. $\frac{s-4}{(s-4)^2+9}$ **Ans.** $e^{4t}\cos 3t$
7. $\frac{16}{(s^2+2s+5)^2}$ **Ans.** $e^{-t}(\sin 2t-2t\cos 2t)$
8. $\frac{1}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-t}(1-\cos t)$

9. $\frac{1}{(s-2)(s^2+1)}$ **Ans.** $\frac{1}{5}e^{2t}-\frac{1}{5}\cos t-\frac{2}{5}\sin t$

10. $\frac{s^2-6s+7}{(s^2-4s+5)^2}$ **Ans.** $te^{2t}\{\cos t-\sin t\}$

11.10 INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$L\left\{\int_0^t f_1(x)*f_2(t-x)\,dx\right\}=F_1(s).F_2(s) \text{ or } \int_0^t f_1(x).f_2(t-x)\,dx=L^{-1}[F_1(s).F_2(s)]$$

Example 34. *Use convolution theorem to evaluate:*

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$ *(U.P., II Semester, 2010)*

Solution. $$\frac{s}{(s^2+4)^2}=\frac{1}{s^2+4}.\frac{s}{s^2+4}$$

Let $$F_1(s)=\frac{1}{s^2+4} \text{ and } F_2(s)=\frac{s}{s^2+4}$$

and $$L^{-1}[F_1(s)]=L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{1}{2}\sin 2t$$

and $$L^{-1}[F_2(s)]=L^{-1}\left(\frac{s}{s^2+4}\right)=\cos 2t$$

According to Convolution Theorem

$$L^{-1}[F_1(s).F_2(s)]=\int_0^t f_1(x).f_2(t-x)dx$$
$$=\int_0^t \frac{1}{2}\sin 2x\cos 2(t-x)dx$$
$$=\frac{1}{4}\int_0^t[\sin(2x+2t-2x)+\sin(2x-2t+2x)]\,dx$$
$$=\frac{1}{4}\int_0^t[\sin 2t+\sin(4x-2t)]dx$$
$$=\frac{1}{4}\left[x\sin 2t-\frac{1}{4}\cos(4x-2t)\right]_0^t$$
$$=\frac{1}{4}\left[t\sin 2t-\frac{1}{4}\cos(4t-2t)+\frac{1}{4}\cos(-2t)\right]$$
$$=\frac{1}{4}\left[t\sin 2t-\frac{1}{4}\cos 2t+\frac{1}{4}\cos 2t\right]=\frac{1}{4}\sin 2t$$ **Ans.**

Example 35. *Use convolution theorem to find the inverse of the function* $\frac{1}{(s^2+a^2)^2}$.

(U.P., II Semester, 2009)

Solution. We know that

$$L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$$

Hence by convolution theorem

$$L^{-1}\frac{1}{(s^2+a^2)}\frac{1}{(s^2+a^2)}=\int_0^t\frac{1}{a}\sin ax\frac{1}{a}\sin a(t-x)\,dx$$
$$=\frac{1}{a^2}\int_0^t\frac{1}{2}[\cos(ax-at+ax)-\cos(ax+at-ax)]\,dx$$
$$\left\{\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]\right\}$$
$$=\frac{1}{2a^2}\int_0^t[\cos(2ax-at)-\cos at)]\,dx$$
$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin(2ax-at)-x\cos at\right]_0^t$$
$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin(2at-at)-t\cos at-\frac{1}{2a}\sin(-at)\right]$$
$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]$$
$$=\frac{1}{2a^2}\left[\frac{2}{2a}\sin at-t\cos at\right]$$
$$=\frac{1}{2a^3}[\sin at-at\cos at]$$ **Ans.**

Example 36. *State convolution theorem and hence find*

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$$ *(Uttarakhand, II Semester, June 2007)*

Solution. Convolution Theorem (See Art 10.23).

Let $L\{f_1(t)\}=F_1(s)$ and Let $L\{f_2(t)\}=F_2(s)$

$$F_1(s)=\frac{1}{(s+2)^2} \text{ and } F_2(s)=\frac{1}{s-2}$$

$$L(e^{-2t})=\frac{1}{s+2}$$
$$L(te^{-2t})=-\frac{d}{ds}\frac{1}{(s+2)}=\frac{1}{(s+2)^2}$$

$$f_1(t)=L^{-1}\left[\frac{1}{(s+2)^2}\right]=te^{-2t}$$

$$f_2(t)=L^{-1}\left[\frac{1}{(s-2)}\right]=e^{2t}$$

According to Convolution Theorem

$$L^{-1}[F_1(s).F_2(s)]=\int_0^t f_1(x)f_2(t-x)dx$$

$$L^{-1}\left[\frac{1}{(s+2)^2(s-2)}\right] = \int_0^t x\,e^{-2x}.e^{2(t-x)}dx = \int_0^t x e^{2t-4x}dx$$

$$=\left[x\frac{e^{2t-4x}}{-4}-\int 1.\frac{e^{(2t-4x)}}{-4}dx\right]_0^t = \left[-\frac{x}{4}e^{2t-4x}+\frac{1}{4}\left\{\frac{e^{(2t-4x)}}{-4}\right\}\right]_0^t$$

$$=\frac{-t}{4}e^{2t-4t}-\frac{1}{16}e^{2t-4t}+\frac{1}{16}e^{2t} = \frac{-t}{4}e^{-2t}-\frac{1}{16}e^{-2t}+\frac{1}{16}e^{2t}$$

$$=\frac{e^{2t}}{16}-\frac{1}{16}e^{-2t}[4t+1]$$

Ans.

Example 37. *Using the Convolution Theorem find*

*

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\},\ a\neq b.$$

(*M.D.U., 2009, U.P. II Semester Summer 2006, 2004*)

Solution. We have, $L(\cos at)=\frac{s}{s^2+a^2}$ and $L(\cos bt)=\frac{s}{s^2+b^2}$

Hence, by the convolution theorem

$$L\left\{\int_0^t \cos ax\cos b(t-x)\,dx\right\}=\frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Therefore,

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\int_0^t \cos ax\cos b(t-x)\,dx$$

$$=\frac{1}{2}\int_0^t\{\cos(ax+bt-bx)+\cos(ax-bt+bx)\}\,dx$$

$$=\frac{1}{2}\int_0^t\cos[(a-b)x+bt]\,dx+\frac{1}{2}\int_0^t\cos[(a+b)x-bt]dx$$

$$=\left[\frac{\sin[(a-b)x+bt]}{2(a-b)}\right]_0^t+\left[\frac{\sin[(a+b)x-bt]}{2(a+b)}\right]_0^t$$

$$=\frac{\sin at-\sin bt}{2(a-b)}+\frac{\sin at+\sin bt}{2(a+b)}$$

$$=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

Ans.

* **Example 38.** *Evaluate* $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$ (*U.P., II Semester, Summer 2002*)

Solution. We know that $L^{-1}\frac{s}{s^2+1}=\cos x$ and $L^{-1}\frac{2}{s^2+2^2}=\sin 2x$

$$L^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right)=\frac{1}{2}L^{-1}\left[\left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^2+4}\right)\right]$$

$$=\frac{1}{2}\int_0^t\sin 2x\cos(t-x)\,dx \quad \text{[By Convolution Th.]}$$

$$=\int_0^t\sin x\cos x\{\cos t\cos x+\sin t\sin x\}\,dx$$

$$=\int_0^t\left[\sin x\cos^2 x\cos t+\sin^2 x\cos x\sin t\right]dx$$

$$=\left[-\frac{\cos^3 x}{3}\cos t+\frac{\sin^3 x}{3}\sin t\right]_0^t$$

$$=-\frac{\cos^4 t}{3}+\frac{\sin^4 t}{3}+\frac{\cos t}{3}=\frac{1}{3}\left[\sin^4 t-\cos^4 t\right]+\frac{\cos t}{3}$$

$$=\frac{1}{3}(\sin^2 t+\cos^2 t)(\sin^2 t-\cos^2 t)+\frac{\cos t}{3}$$

$$=\frac{1}{3}(\sin^2 t-\cos^2 t)+\frac{\cos t}{3} = -\frac{1}{3}\cos 2t+\frac{\cos t}{3}$$

$$=\frac{1}{3}(\cos t-\cos 2t)$$

Ans.

Example 39. *Obtain* $L^{-1}\frac{1}{s(s^2+a^2)}$.

Solution. $L^{-1}\frac{1}{s}=1$ and $L^{-1}\frac{1}{s^2+a^2}=\frac{\sin at}{a}$.

$$L^{-1}\{F_1(s).F_2(s)\}=\int_0^t f_1(t)f_2(t-x)dx \quad \text{(Convolution Theorem)}$$

Hence by the Convolution Theorem

$$L^{-1}\left[\frac{1}{s}.\frac{1}{s^2+a^2}\right]=\int_0^t\frac{\sin a(t-x)}{a}dx=\left[\frac{-\cos(at-ax)}{-a^2}\right]_0^t$$

$$=\frac{1}{a^2}[1-\cos at]$$

Ans

Example 40. *Using Convolution Theorem, prove that*

*

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]=\frac{t^2}{2}+\cos t-1$$

(*U.P., II Semester, Summer 2005*)

Solution. We know that,

$$L^{-1}\left\{\frac{1}{s^3}\right\}=\frac{t^2}{2!}$$

$$L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$$

Using Convolution Theorem,

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t \frac{(t-x)^2}{2!}\sin x\,dx$$

$$= \frac{1}{2}\int_0^t \left(t^2+x^2-2tx\right)\sin x\,dx$$

$$= \frac{1}{2}\left[\left(t^2+x^2-2tx\right)(-\cos x) - \int(2x-2t)(-\cos x)\,dx\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+x^2-2tx)(-\cos x) + 2\int(x-t)\cos x\,dx\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+x^2-2tx)(-\cos x) + 2(x-t)\sin x + 2\cos x\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+t^2-2t^2)(-\cos t) + 0 + 2\cos t + t^2\cos 0 - 2\cos 0\right]$$

$$= \frac{1}{2}[2\cos t + t^2 - 2] = \cos t + \frac{t^2}{2} - 1 = \frac{t^2}{2} + \cos t - 1$$ **Ans.**

EXERCISE 11.8

Obtain the Inverse Laplace Transform by convolution.

1. $\frac{s^2}{\left(s^2+a^2\right)^2}$ **Ans.** $\frac{1}{2}t\cos at + \frac{1}{2a}\sin at$
2. $\frac{1}{\left(s^2+1\right)^3}$ **Ans.** $\frac{1}{8}\left[\left(3-t^2\right)\sin t - 3t\cos t\right]$
3. $\frac{s}{\left(s^2+a^2\right)^2}$ **Ans.** $\frac{t\sin at}{2a}$
4. $\frac{1}{s^2\left(s^2-a^2\right)}$ **Ans.** $\frac{1}{a^3}[-at+\sinh at]$
5. $\frac{1}{(s+1)\left(s^2+1\right)}$ **Ans.** $\frac{1}{2}\left(\cos t - \sin t - e^{-t}\right)$

11.11 HEAVISIDE INVERSE FORMULA OF $\frac{F(s)}{G(s)}$

If $F(s)$ and $G(s)$ be two polynomials in S. The degree of F(s) is less than that of G (s). Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be n roots of the equation $G(s) = 0$

Inverse Laplace formula of $\frac{F(s)}{G(s)}$ is given by

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t}$$

Example 41. *Find* $L^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\}$.

Solution. Let $F(s) = 2s^2+5s-4$

and $G(s) = s^3+s^2-2s = s(s^2+s-2) = s(s+2)(s-1)$

$G'(s) = 3s^2+2s-2$

$G(s) = 0$ has three roots, 0, 1, – 2

$\Rightarrow$ $\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = -2$

By Heaviside Inverse formula

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t}$$

$$= \left\{\frac{F(\alpha_1)}{G'(\alpha_1)}\right\}e^{t\alpha_1} + \frac{F(\alpha_2)}{G'(\alpha_2)}e^{t\alpha_2} + \frac{F(\alpha_3)}{G'(\alpha_3)}e^{t\alpha_3} = \frac{F(0)}{G'(0)}e^0 + \frac{F(1)}{G'(1)}e^t + \frac{F(-2)}{G'(-2)}e^{-2t}$$

$$= \frac{-4}{-2}e^0 + \frac{3}{3}e^t + \frac{(-6)}{(6)}e^{-2t} = 2 + e^t - e^{-2t}$$ **Ans.**

Example 42. *Find* $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$ *(U.P. II Semester, 2004)*

Solution. Let $F(s) = 2s^2-6s+5$

$G(s) = s^3-6s^2+11s-6 = (s-1)(s-2)(s-3)$

$G(s) = 0$ has three roots, 1, 2, 3.

$\Rightarrow$ $\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 3$

$G'(s) = 3s^2 - 12s + 11$

By Heaviside Inverse formula, we have $L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{t\alpha_i}$

$$L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)}e^{t\alpha_1} + \frac{F(\alpha_2)}{G'(\alpha_2)}e^{t\alpha_2} + \frac{F(\alpha_3)}{G'(\alpha_3)}e^{t\alpha_3}$$

$$= \frac{F(1)}{G'(1)}e^t + \frac{F(2)}{G'(2)}e^{2t} + \frac{F(3)}{G'(3)}e^{3t} = \frac{(1)}{(2)}e^t + \frac{(1)}{(-1)}e^{2t} + \frac{(5)}{(2)}e^{3t} = \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$ **Ans.**

EXERCISE 11.9

Using Heaviside expansion formula, find the Inverse Laplace Transform of the following :

1. $\frac{s-1}{s^2+3s+2}$ **Ans.** $-2e^{-t}+3e^{-2t}$
2. $\frac{s}{(s-1)(s-2)(s-3)}$ **Ans.** $\frac{1}{2}e^t - 2e^{2t} + \frac{3}{2}e^{3t}$
3. $\frac{2s+3}{(s-2)(s-3)(s-4)}$ **Ans.** $\frac{7}{2}e^{2t} - 9e^{3t} + \frac{11}{2}e^{4t}$
4. $\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ **Ans.** $2e^{-t} + 5\cdot e^{2t} - \frac{3}{2}e^{\frac{t}{2}}$